Generalized Clifford Algebras and the Last Fermat Theorem

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Abstract

One shows that the Last Fermat Theorem is equivalent to the statement that all rational solutions $x^k + y^k = 1$ of equation ($k \geq 2$) are provided by an orbit of rationally parametrized subgroup of a group preserving k–ubic form. This very group naturally arrises in the generalized Clifford algebras setting [1].

I. The stroboscopic motion of the independent oscilatory degree of freedom is given by iteration of the "classical map" matrix

$$L(\Delta) = \frac{1}{1+\Delta^2} \begin{pmatrix} 1-\Delta^2 & -2\Delta \\ 2\Delta & 1-\Delta^2 \end{pmatrix} \quad \Delta \in \bar{\mathbf{Q}} = \mathbf{Q} \cup \{\infty\} \tag{1}$$

(see [2] and references therein).
$L(\Delta)$ of (1) provides the rational parametrization of the unit circle obtained via stereographic projection composed with $\pi/2$–rotation represented by an $\mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ imaginary unit matrix.
The set $\mathbf{SO}(2;\mathbf{Q}) = \{L(\Delta);\ \Delta \in \bar{\mathbf{Q}}\}$ is the well known group. Ocasionally it is a refine exercise to prove that $L(\Delta_1)L(\Delta_2) = L(\Delta)$, where $\Delta_1, \Delta_2 \in \bar{\mathbf{Q}}$

$$\Delta = \frac{\Delta_1 + \Delta_2}{1 - \Delta_1\Delta_2} = \frac{\left(1+\Delta_1^2\right)\left(1+\Delta_2^2\right) - \left(1-\Delta_1^2\right)\left(1-\Delta_2^2\right) + 4\Delta_1\Delta_2}{2\left[\Delta_1\left(1-\Delta_2^2\right) + \Delta_2\left(1-\Delta_1^2\right)\right]} \tag{2}$$

with special cases such as $L(1)L(1) = L(\infty) = -\mathbf{1}$ or $L(\Delta)L(-\Delta) = L(0) = \mathbf{1}$ included (for the last one use d'Hospital rule).
One is tempted to call the group

$$\mathbf{O}(2;\bar{\mathbf{Q}}) = \mathbf{SO}(2;\bar{\mathbf{Q}}) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{SO}(2;\bar{\mathbf{Q}})$$

the 2–Fermat group as it preserves quadratic form

$$x^2 + y^2 = 1 \qquad x, y \in \mathbf{Q} \tag{3}$$

and even more.
Observation: $\mathbb{Q}(2;\bar{\mathbf{Q}})$ group acts transitively on the set of all rational solutions of (3).
Proof: For any two solutions $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix}$ one easily finds $A \in \mathbb{Q}(2;\bar{\mathbf{Q}})$ such that $\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. For example: let $x \neq -x_0$ and let $y \neq -y_0$; then $A = L(\Delta)$,

$$\Delta = \frac{x_0 y - x y_0}{x_0(x_0+x) + y_0(y_0+y)}$$

The shape of formula for Δ depends on the way one chooses to find it out. One way is just straightforward calculation. The other is based on the observation that for $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\Delta = \frac{y}{x+1}$. Hence for any $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ & $\begin{pmatrix} x \\ y \end{pmatrix}$ the corresponding Δ is being found due to the obvious identity $L(\Delta) \equiv L\left(\frac{y}{x+1}\right) L\left(-\frac{y_0}{x_0+1}\right)$. That way we arrive at the intriguing identity valid for all **solutions** of (3) i.e.

$$\frac{x_0 y - x y_0}{x_0(x_0+x) + y_0(y_0+y)} \equiv \frac{x_0 y - x y_0 + y - y_0}{x_0(x_0+x) + y_0(y_0+y) + x + x_0} \tag{4}$$

Conclusion: It is enough to start with trivial solution $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of (3). All others are obtained as elements of the corresponding orbit of $\mathbb{Q}(2;\bar{\mathbf{Q}})$ i.e. 2–Fermat group.
Remark: An iteration of $L(\Delta)$, i.e. $L(\Delta) \to L^2(\Delta) \to ... L^k(\Delta) \to ...$ provides us with stroboscopic motion in one oscilatory degree of freedom which in view of (2) is chaotic; it is in a sense – "number theoretic" – chaotic. (For the relation to Fibonacci–like sequences – see [2])

II. Consider now

$$x^k + y^k = 1 \quad k \geq 3,\ n \in \mathbf{N} \tag{5}$$
$$\text{where} \quad x, y \in \mathbb{C}.$$

Denote by $\mathbb{k}(2;\mathbb{C})$ the group of all linear transformations preserving this k–ubic form [1] related to generalized Clifford algebras [1]. Of course starting from any – say trivial solution $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of (5), the orbit $\mathbb{k}(2;\mathbb{C})$ would provide us with a family of other solutions. Starting with another, nontrivial solution

$$\begin{pmatrix} x \\ \sqrt[k]{1-x^k} \end{pmatrix} \quad x \neq 1 \quad x \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

we get – for each another x (not belonging to the precedent orbit!) a new orbit of solutions. Evidently the set of all complex solutions of $x^k + y^k = 1$ has the structure of the sum of disjoint orbits of $\mathbb{k}(2;\mathbb{C})$. In this connection note that the relation between two solutions belonging to different orbits must be nonlinear.
According to K. Morinaga and T. Nono [3, 1]

$$\mathbb{k}(n;\mathbb{C}) = \left\{ \omega^l \delta_{i,\sigma(j)};\ l \in \mathbf{Z}_k,\ \sigma \in \mathbf{S}_n \right\} \quad k \geq 3 \tag{6}$$

where $\omega = \exp\left\{\frac{2\pi i}{k}\right\}$. Naturally $|\textcircled{k}(n;\mathbb{C})\,| = k^n n!$, hence every "$k$–Fermat group" orbit of solutions of (5) counts $2k^2$ elements.

One readily notices that the orbit $\textcircled{k}(2;\mathbb{C})\begin{pmatrix}1\\0\end{pmatrix}$ does not exhibit any nontrivial rational solution, as the k–Fermat group, $k \geq 3$ i.e. $\textcircled{k}(n;\mathbb{C})$ contains the only one **rationaly** parametrized subgroup, i.e. the matrix permutation subgroup $\simeq \mathbf{S}_n$.

Thus we arrive at the

Conclusion: The Last Fermat Theorem is equivalent to the statement, that all avaiable rational solutions of $x^k + y^k = 1 \;\; k \geq 2$ are provided by the orbit $\textcircled{k}(2;\bar{\mathbf{Q}})\begin{pmatrix}1\\0\end{pmatrix}$; $\;\textcircled{k}(n;\mathbf{Q}) \subset \textcircled{k}(n;\mathbb{C})$.

One is evidently tempted to conjecture the "corresponding Last Fermat Theorem" concerning $\textcircled{k}(n;\bar{\mathbf{Q}}) \;\; n > 2$ group. Hence n–hypothesis. Let $n \geq 2$, then

$$x_1^k + x_2^k + ... + x_n^k = 1$$

has no rational solutions for $k \geq 3$, except for trivial ones, i.e. $x_s = 0, \pm 1, \; s = 1, ..., n$.

This is however obviously **false**, since for each x_1 – natural and k – odd numbers it is easy to find natural n and $x_2, ..., x_n$ such that equation is true. Anyhow quadratic forms for $n = 2$ (appropiate to associate oscilations with!) seem to be the only ones among k–ubic forms ($k \geq 2, \; n = 2$) that would provide us with nontrivial stroboscopic motion by group element iteration as outlined in [2].

Remark 1: k–ubic forms of (1,1) signature as well as corresponding generalized Clifford algebras are at hand [1], hence the "2–hypothesis" equipped with (1,1) signature is easy to formulate; namely: Let Q be a k–ubic form of (1,1) signature. Let $\vec{x} \in \mathbf{Q}^2$; then the all solutions of $Q(\vec{x}) = \mathbf{1}$ are given by the orbit

$$\textcircled{k}(1,1;\mathbf{Q})\begin{pmatrix}1\\0\end{pmatrix}.$$

(This is of course equivalent to the (2,0) signature case)

For the sake of examplification take $k = 2, \; n = 2$. Then

$$\textcircled{2}(1,1;\bar{\mathbf{Q}}) = \mathbf{S}\textcircled{2}(1,1;\bar{\mathbf{Q}}) \cup \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{S}\textcircled{2}(1,1;\bar{\mathbf{Q}})$$

where

$$\mathbf{S}\textcircled{2}(1,1;\bar{\mathbf{Q}}) \equiv \{\tilde{L}(\Delta);\; \Delta \in \bar{\mathbf{Q}}\};\;\; \tilde{L}(\Delta) \equiv \frac{1}{1-\Delta^2}\begin{pmatrix}1+\Delta^2 & 2\Delta\\ 2\Delta & 1+\Delta^2\end{pmatrix}.$$

It is then easy to see, that

Observation: $\textcircled{2}(1,1;\bar{\mathbf{Q}})$ group acts transitively on the set of all rational solutions of $x^2 - y^2 = 1$.

Proof: For any two solutions $\begin{pmatrix}x_0\\y_0\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}$ one easily finds $L(\Delta) \in \textcircled{2}(1;1;\bar{\mathbf{Q}})$ such that $\begin{pmatrix}x\\y\end{pmatrix} = L(\Delta)\begin{pmatrix}x_0\\y_0\end{pmatrix}$. For example: let $x \neq -x_0$ and $y \neq -y_0$; then one has the following identity

$$\Delta = \frac{xy_0 - yx_0}{x(x_0+x) + y(y_0+y)} \equiv \frac{x_0 y - xy_0 + y - y_0}{x_0(x_0+x) - y_0(y_0+y) + x + x_0} \tag{7}$$

(7) is analogous to (4) i.e. it is valid on the set of solutions of "hiperbolic" Fermat $n = 2$ equation

$$x^2 - y^2 = 1\;;\;\; x, y \in \bar{\mathbf{Q}}$$

The formula analogous to (2) has the form:

$$\Delta = \frac{\Delta_1 + \Delta_2}{1 + \Delta_1 \Delta_2} = \frac{\left(1 - \Delta_1^2\right)\left(1 - \Delta_2^2\right) - \left(1 + \Delta_1^2\right)\left(1 + \Delta_2^2\right) - 4\Delta_1\Delta_2}{2\left[\Delta_1\left(1 + \Delta_2^2\right) + \Delta_2\left(1 + \Delta_1^2\right)\right]} \tag{8}$$

where

$$L(\Delta) \equiv L(\Delta_1)\, L(\Delta_2) \ ; \quad L(\Delta_1)\,,\ L(\Delta_2) \in \mathbf{S}\mathbf{O}(1,1;\bar{\mathbf{Q}})$$

with special cases such as $\tilde{L}(\Delta)\tilde{L}\left(-\frac{1}{\Delta}\right) = \tilde{L}(\infty) = -\mathbf{1}$ or $\tilde{L}(\Delta)\tilde{L}(-\Delta) = \tilde{L}(0)$ included.
Remark 2: We suggest relevance of hyperbolic functions of k–th order [4] in relations between LFT and generalized Clifford algebras (as used to linearize k–ubic forms in a Dirac way).

References

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